

LARGE GAPS IN THE IMAGE OF THE EULER'S FUNCTION

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ABSTRACT. The aim of this note is to provide an upper bound of the number of positive integers $\leq x$ which can be written as $\varphi(n)$ for some positive integer n , where φ stands for the Euler's function. The order of magnitude of this estimate, which is roughly $x/\sqrt[4]{\ln x}$, implies that the set of Euler's values contains arbitrarily large gaps.

1. INTRODUCTION

Let φ be the Euler's function, so that $\varphi(n)$ stands for the number of integers smaller or equal than n and coprime with n , and denote with V the set of Euler's values, that is, the set of positive integers which can be written as $\varphi(n)$ for some positive integer n . Then, we show that the set V contains arbitrarily large gaps, viz.

$$\limsup_{n \in \mathbf{N}} v_{n+1} - v_n = \infty,$$

where (v_1, v_2, \dots) represents the sequence of elements of V in increasing order. Let also $V(x)$ be the number of elements of V smaller than x . The existence of arbitrarily large gaps in the set V follows by:

Theorem 1. *There exists a positive constant c such that for all $x \geq 2$ we have*

$$V(x) \leq c \frac{x}{(\ln x)^{.2587966}}.$$

Indeed, it implies that $V(x) = o(x)$, hence $v_{n+1} - v_n$ cannot be bounded. It is worth noticing that Theorem 1 is just a refinement of the estimate given by Pillai [4], where he obtained an upper bound with order of magnitude $x/(\ln x)^{\ln 2/e}$.

As far as the Euler's function is bijective on primes, it is immediate that $V(x)$ is greater than the number of primes smaller than or equal to x . Then, according to the Prime Number Theorem (or to the elementary bounds provided by Chebyshev, see e.g. [1]), $V(x)$ has to be a order of magnitude greater than or equal to $x/\ln x$. A natural question arises: what is the “correct” exponent t such that the order of magnitude of $V(x)$ is exactly $x/(\ln x)^t$? In its seminal paper [2], Erdős proved that the answer has to be 1, meaning that the φ -values in $[1, x]$ are so exceptional that they are just $x/(\ln x)^{1+o(1)}$. Further improvements can be found in [5] and [3]. At the moment, we still do not know if a natural asymptotic formula exists.

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2. PREPARATIONS

Here and later, \mathbf{N} , \mathbf{P} , and \mathbf{R} stand for the set of positive integers, primes, and reals, respectively. At this point, we are going to prove some preliminary lemmas: all of them are standard results in number theory. Therefore, the reader who just wants to read the proof of Theorem 1 can skip directly to Section 3.

Lemma 2. Let μ represent the Möbius function, which is the arithmetical function defined by $\mu(1) = 1$, $\mu(n) = (-1)^k$ if $n \geq 2$ is squarefree, where k is the number of distinct prime factors of n , and $\mu(n) = 0$ otherwise. Let also m be a positive integer and N a non-empty subset of \mathbf{N} . Then, the number of integers which belong to N , smaller than or equal to x , and coprime with m is equal to

$$\sum_{d|m} \mu(d) |N \cap d\mathbf{N} \cap [1, x]|.$$

Proof. It is easy to see that the arithmetical function $n \mapsto \sum_{d|n} \mu(d)$ is multiplicative, so that it is equal to 1 if $n = 1$ and 0 otherwise (indeed, it is enough to verify it for the prime powers); then the required sum is equal to

$$\sum_{\substack{n \in N \cap [1, x], \\ \gcd(n, m) = 1}} 1 = \sum_{n \in N \cap [1, x]} \left(\sum_{d|\gcd(n, m)} \mu(d) \right).$$

Let us evaluate this value by double counting: fix a positive integer d which divides $\gcd(n, m)$. In particular the set of possible values of d will be a subset of divisors of m , so that the element $\mu(d)$ will be counted exactly $|\{n \in N \cap [1, x] : d | n\}|$ times. It follows that the required number can be rewritten as

$$\sum_{d|m} \left(\mu(d) \sum_{n \in N \cap [1, x], d|n} 1 \right),$$

which is equivalent to the claim. ■

This first result allows us to deduce the rather famous formula for the computation of $\varphi(n)$. Indeed, setting $x = n$ and N equal to set of positive integers $\leq n$, we obtain that

$$\varphi(n) = \sum_{d|n} \mu(d) |N \cap d\mathbf{N} \cap [1, n]| = n \sum_{d|n} \frac{\mu(d)}{d} = n \prod_{p|n} \left(1 - \frac{1}{p} \right).$$

Lemma 3. (Abel summation). Let $(\lambda_n)_{n \in \mathbf{N}}$ be a strictly increasing and unbounded sequence of positive reals, $(a_n)_{n \in \mathbf{N}}$ a sequence of complex numbers, and f a differentiable complex-valued function defined on positive reals. Define also $\alpha(x) = \sum_{\lambda_n \leq x} a_n$ for all reals x greater than λ_1 . Then

$$\sum_{\lambda_n \leq x} a_n f(\lambda_n) = \alpha(x) f(x) - \int_{\lambda_1}^x \alpha(t) f'(t) dt.$$

Proof. Define for convenience $\lambda_0 = \alpha(0) = 0$. Then for all positive integers m

$$\begin{aligned} \sum_{1 \leq n \leq m} a_n f(\lambda_n) &= \sum_{1 \leq n \leq m} (\alpha(\lambda_n) - \alpha(\lambda_{n-1})) f(\lambda_n) \\ &= \alpha(\lambda_m) f(\lambda_m) - \sum_{1 \leq n \leq m-1} \alpha(\lambda_n) (f(\lambda_{n+1}) - f(\lambda_n)). \end{aligned}$$

At this point, let x be a real greater than λ_1 , and define m the greatest integer such that $\lambda_m \leq x$, which exists by assumption. Considering that $\sum_{\lambda_n \leq x} a_n f(\lambda_n)$ is equal to $\sum_{1 \leq n \leq m} a_n f(\lambda_n)$ and that the function α is constant in $[\lambda_n, \lambda_{n+1})$, then

$$\begin{aligned} \sum_{\lambda_n \leq x} a_n f(\lambda_n) &= \alpha(\lambda_m) f(\lambda_m) - \sum_{1 \leq n \leq m-1} \alpha(\lambda_n) (f(\lambda_{n+1}) - f(\lambda_n)) \\ &= \alpha(\lambda_m) f(\lambda_m) - \sum_{1 \leq n \leq m-1} \alpha(\lambda_n) \int_{\lambda_n}^{\lambda_{n+1}} f'(t) dt \\ &= \alpha(\lambda_m) f(\lambda_m) - \sum_{1 \leq n \leq m-1} \int_{\lambda_n}^{\lambda_{n+1}} \alpha(t) f'(t) dt. \end{aligned}$$

Hence it turns out that this sum is equal to

$$\alpha(\lambda_m) f(\lambda_m) - \int_{\lambda_1}^{\lambda_m} \alpha(t) f'(t) dt,$$

which is equivalent to the claim. ■

It is worth noticing that, in the setting of Stieltjes integral, the summation takes the innocuous form of partial integration, that is why this result is commonly known as “partial summation.” Indeed, under the assumption of Lemma 3, the required sum can be directly rewritten as

$$\int_{\lambda_1}^x f(t) d\alpha(t) = [\alpha(t)f(t)]_{\lambda_1}^x - \int_{\lambda_1}^x \alpha(t) f'(t) dt.$$

For convenience, from here later, we are going to use the Bachmann-Landau notations: given real-valued functions $f, g: \mathbf{R} \rightarrow \mathbf{R}$, the big-Oh $f(x) = \mathcal{O}(g(x))$ stands for the existence of a constant c such that (the absolute value of) their ratio is upper bounded by c whenever x is sufficiently large, that is $\limsup_{x \rightarrow \infty} |f(x)/g(x)| \leq c$. Moreover, the little-oh $f(x) = o(g(x))$ means that $|f(x)|$ is definitively arbitrarily smaller than $|g(x)|$, i.e. $\lim_{x \rightarrow \infty} |f(x)/g(x)| = 0$.

Lemma 4. Let k be a positive integer and c a positive constant. Then the number of integers n smaller than x with exactly k distinct prime factors is smaller than cx whenever x is sufficiently large.

Proof. For each positive integer k , let $\varrho_k(x)$ be the number of positive integers n smaller than or equal to x with exactly k distinct prime factors. In terms of Bachmann-Landau notations, the statement is equivalent to $\varrho_k(x) = o(x)$ and it would be sufficient to prove that

$$\varrho_k(x) = \mathcal{O}\left(\frac{x}{\ln x} (\ln \ln x)^{k-1}\right). \quad (1)$$

Let us show this claim by induction, starting from the case $k = 1$. For each prime p smaller than or equal to x , the number of powers of p in $[1, x]$ is exactly $\lfloor \log_p(x) \rfloor$. Then, summing over all primes we obtain

$$\varrho_1(x) = \sum_{p \leq x} \left\lfloor \frac{\ln x}{\ln p} \right\rfloor = \mathcal{O} \left(\ln x \sum_{p \leq x} \frac{1}{\ln p} \right).$$

Setting $f(t) = \frac{1}{\ln t}$ in Lemma 3, we evaluate the sum $\sum_{p \leq x} \frac{1}{\ln p}$ so that

$$\varrho_1(x) = \mathcal{O} \left(\pi(x) + \ln x \int_2^x \frac{\pi(t)}{t \ln^2 t} dt \right),$$

where $\pi(t)$ stands for the number of primes $\leq t$. Considering that $\pi(t) = \mathcal{O} \left(\frac{t}{\ln t} \right)$ by the Prime Number Theorem, we have that the argument of the integral is equal to $\mathcal{O} \left(\frac{1}{\ln^3 t} \right) = \mathcal{O} \left(\frac{1}{\ln^2 t} \right) = \mathcal{O} \left(\frac{d}{dt} \frac{t}{\ln^2 t} \right)$. Therefore

$$\varrho_1(x) = \mathcal{O}(\pi(x)) + \mathcal{O} \left(\ln x \int_2^x \frac{d}{dt} \frac{t}{\ln^2 t} dt \right) = \mathcal{O}(\pi(x)).$$

At this point, suppose that the claim (1) holds for a positive integer k , and let us prove that $\varrho_{k+1}(x) = \mathcal{O} \left(\frac{x}{\ln x} (\ln \ln x)^k \right)$. For each positive integer α_0 smaller than $\ln x$, consider the numbers of the form $p_0^{\alpha_0} \cdot (p_1^{\alpha_1} \cdots p_k^{\alpha_k})$ not greater than x , such that $p_1 < \cdots < p_k$, and $p_0^{\alpha_0+1} \leq x$. Notice that each number $n \leq x$ with $\omega(n) = k+1$ can be expressed in such form at least $k+1$ times, depending on the position of the first factor. It follows that

$$(k+1)\varrho_{k+1}(x) \leq \sum_{n \in \mathbf{N} \cap [1, \ln x]} \sum_{p^{n+1} \leq x} \varrho_k \left(\frac{x}{p^n} \right),$$

and in particular

$$\varrho_{k+1}(x) = \mathcal{O} \left(\sum_{n \in \mathbf{N}} \sum_{p^{n+1} \leq x} \varrho_k \left(\frac{x}{p^n} \right) \right).$$

Since we assumed that $\varrho_k(x) = \mathcal{O} \left(\frac{x}{\ln x} (\ln \ln x)^{k-1} \right)$, we obtain that

$$\varrho_{k+1}(x) = \mathcal{O} \left(x (\ln \ln x)^{k-1} \sum_{n \in \mathbf{N}} \sum_{p \leq x^{1/(n+1)}} \frac{1}{p^n \ln(x/p^n)} \right).$$

Now, the key observation is that the main contribution of the summation comes from the index $n = 1$. For all positive reals a, b such that $a \geq 2b$ we have $\frac{1}{a-b} \leq \frac{1}{a} + \frac{2b}{a^2}$, therefore

$$\sum_{p^2 \leq x} \frac{1}{p(\ln x - \ln p)} = \mathcal{O} \left(\frac{1}{\ln x} \sum_{p \leq \sqrt{x}} \frac{1}{p} \right) + \mathcal{O} \left(\frac{1}{\ln^2 x} \sum_{p \leq \sqrt{x}} \frac{\ln p}{p} \right).$$

On the one hand, setting $f(x) = \frac{1}{x}$ in Lemma 3, we obtain

$$\sum_{p \leq \sqrt{x}} \frac{1}{p} \leq \frac{\pi(x)}{x} + \int_2^x \frac{\pi(t)}{t^2} dt = \mathcal{O} \left(\frac{1}{\ln x} \right) + \mathcal{O} \left(\int_2^x \frac{d}{dt} \ln \ln t dt \right) = \mathcal{O}(\ln \ln x).$$

On the other hand, with the same argument, setting $f(x) = \frac{\ln x}{x}$ we get

$$\sum_{p \leq \sqrt{x}} \frac{\ln p}{p} \leq \frac{\pi(x)}{x} + \int_2^x \frac{\ln t - 1}{t \ln t} dt = \mathcal{O}(\ln x).$$

Notice that last two upper bounds follow from the first and second Mertens' theorems (see, for example, [1]). Putting these results together, we conclude that

$$\varrho_{k+1}(x) = \mathcal{O}\left(\frac{x}{\ln x} (\ln \ln x)^k\right) + \mathcal{O}\left(x (\ln \ln x)^{k-1} \sum_{n \geq 2} \sum_{p^{n+1} \leq x} \frac{1}{p^n \ln(x/p^n)}\right).$$

Since that condition $p^{n+1} \leq x$ is equivalent to $\ln(x/p^n) \geq \frac{\ln x}{n+1}$, we have also

$$\begin{aligned} \sum_{n \geq 2} \sum_{p^{n+1} \leq x} \frac{1}{p^n \ln(x/p^n)} &= \mathcal{O}\left(\frac{1}{\ln x} \sum_{n \geq 2} \sum_{p^{n+1} \leq x} \frac{n}{p^n}\right) = \mathcal{O}\left(\frac{1}{\ln x} \sum_{n \geq 2} \sum_{m \geq 2} \frac{n}{m^n}\right) \\ &= \mathcal{O}\left(\frac{1}{\ln x} \sum_{n \geq 2} \frac{n^2}{2^n}\right) = \mathcal{O}\left(\frac{1}{\ln x}\right). \end{aligned}$$

This is enough to conclude that (1) holds also for $k+1$, completing the proof. \blacksquare

Observe that the existence of a constant c_k such that $\varrho_{k+1}(x) \leq c_k \frac{x}{\ln x} (\ln \ln x)^k$ assumes that k is a fixed positive integer, not depending on x . As far as we are interested in the case $c_k = c_k(x)$, we can repeat the above proof to obtain

$$\varrho_k(x) = \mathcal{O}\left(\frac{x}{\ln x} \cdot \frac{(\ln \ln x)^{k-1}}{(k-1)!}\right). \quad (2)$$

It implies that, choosing k as function of x , an upper bound of $\varrho_k(x)$ needs an estimate of the order of magnitude of the factorial $(k-1)!$.

Lemma 5. (Stirling formula). There exists a constant c such that

$$\ln n! = n \ln n - n + \ln \sqrt{n} + c + \mathcal{O}(1/n).$$

Proof. Defining $\text{frac}(t)$ the fraction part of t , that is $t - \lfloor t \rfloor$, the value $\ln n!$, according to Lemma 3, is equal to

$$\sum_{m \leq n} \ln m = n \ln n - \int_1^n \frac{\lfloor t \rfloor}{t} dt = n \ln n - (n-1) + \int_1^n \frac{\text{frac}(t)}{t} dt.$$

Let g be the map defined by $x \mapsto \frac{1}{2} (\lfloor x \rfloor + \text{frac}^2(x))$. Then g is continuous and derivable in each non-integer point x . It implies that $g'(x)$ is exactly $\text{frac}(x)$, and integrating by parts we obtain

$$\int_1^n \frac{\text{frac}(t)}{t} dt = \left[\frac{g(t)}{t} \right]_1^n + \int_1^n \frac{g(t)}{t^2} dt = \int_1^n \frac{t}{2t^2} dt + \int_1^n \frac{\text{frac}^2(t) - \text{frac}(t)}{2t^2} dt.$$

Moreover, the last integral is convergent to some constant c , indeed

$$\lim_{n \rightarrow \infty} \int_1^n \left| \frac{\text{frac}^2(t) - \text{frac}(t)}{t^2} \right| dt = \mathcal{O}\left(\sum_{m \leq n} \frac{1}{m^2}\right) = \mathcal{O}(1).$$

At this point, the proof is complete, noticing that

$$\int_1^n \frac{\text{frac}^2(t) - \text{frac}(t)}{t^2} dt = c + \int_n^\infty \frac{\text{frac}^2(t) - \text{frac}(t)}{t^2} dt = c + \mathcal{O}(1/n).$$

■

Everything is finally ready to prove Theorem 1.

3. PROOF OF THE THEOREM 1

Proof. Let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ be an integer greater than 1, then $(p_1 - 1) \cdots (p_k - 1)$ divides $\varphi(n)$ according to Lemma 2. It implies that, in the worst case, $\varphi(n)$ is divisible by 2^{k-1} .

Let N_k be the set of positive integers with at most k distinct prime factors, and M_k its complementary set, so that if n belongs to M_k then 2^k divides $\varphi(n)$. Considering that if X, Y, Z are sets such that $X = Y \cup Z$ then $|X| \leq |Y| + |Z|$, we get

$$\begin{aligned} V(x) &\leq |\varphi(N_k) \cap [1, x]| + |\varphi(M_k) \cap [1, x]| \\ &\leq |N_k \cap [1, x]| + |\varphi(M_k) \cap [1, x]| \\ &\leq \varrho_1(x) + \dots + \varrho_k(x) + 2^{-k}x. \end{aligned}$$

According to Lemma 4 and the estimate (2), the inequality $\varrho_k(x) \leq \varrho_{k+1}(x)$ holds whenever k is smaller than $\ln \ln x$. Define then $k = k(x) = \lceil c \ln \ln x \rceil$ for some positive real c smaller than 1, and let us try to minimize the right hand side of the above inequality. Clearly, we would have

$$V(x) \leq k \varrho_k(x) + 2^{-k}x.$$

On the one hand, we have that $2^{-k}x$ has order of magnitude $\frac{x}{(\ln x)^{c \ln 2}}$, implying that the greater c is, the smaller its value will be. On the other hand, we have also

$$k \varrho_k(x) = \mathcal{O}(k \varrho_{k+1}(x)) = \mathcal{O}\left(\frac{x}{\ln x} \cdot \frac{(\ln \ln x)^k}{(k-1)!}\right).$$

Taking the logarithm at each side we get

$$\ln(k \varrho_k(x)) = \ln x - \ln \ln x + k \ln \ln \ln x - \ln(k-1)! + \mathcal{O}(1).$$

Since we are taking care here only in addends on order of magnitude at least $\ln \ln x$, the term $\ln(k-1)!$ can be substituted with $\ln k!$ as far as their difference is just $\mathcal{O}(\ln \ln \ln x)$. According to the approximation provided in Lemma 5 we obtain

$$\begin{aligned} \ln(k \varrho_k(x)) &= \ln x - \ln \ln x + k \ln \ln \ln x - k \ln k + k + \mathcal{O}(\ln \ln \ln x) \\ &= \ln x - \ln \ln x(1 - c + c \ln c) + \mathcal{O}(\ln \ln \ln x). \end{aligned}$$

This is enough to conclude that for all c in $(0, 1)$ we have

$$V(x) = \mathcal{O}\left(\frac{x}{(\ln x)^{1-c+c \ln c}}\right) + \mathcal{O}\left(\frac{x}{(\ln x)^{c \ln 2}}\right) = \mathcal{O}\left(\frac{x}{(\ln x)^{\min\{1-c+c \ln c, c \ln 2\}}}\right).$$

Since the function $c \mapsto (1 - c + c \ln c)$ is strictly decreasing in the interval $(0, 1)$, the smaller c is, the smaller $\frac{x}{(\ln x)^{1-c+c \ln c}}$ will be. It implies that the best upper bound estimate is obtained in the case that $1 - c + c \ln c = c \ln 2$, that is when c is equal to $c^* \approx .3733646177$.

We can finally conclude that

$$V(x) = \mathcal{O}\left(\frac{x}{(\ln x)^{c^* \ln 2}}\right) = \mathcal{O}\left(\frac{x}{(\ln x)^{.2587966}}\right),$$

which is the optimal result in line with the original Pillai's estimate. ■

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